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CHARACTERS OF WREATH PRODUCTS AND SOME APPLICATIONS TO REPRESENTATION THEORY AND COMBINATORICS

Adalbert KERBER
Rhein.-Westf. Technische Hochschule, Aachen, Federal Republic of Germany

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The salient point arising out of a consideration of some seemingly independent topics in representation theory, combinatorics and the theory of numerical polynomials turns out to be a result involving characters of representations of wreath products. The topics are: symmetrized inner products of representations, irreducible characters of wreath products, Frobenius' formula for the irreducible ordinary characters of symmetric groups, the Pólya–Redfield theory of enumeration under group action in combinatorics and results of Rudvalis and Snapper that certain polynomials arising from generalized cycle-indices of permutation groups are numerical.

1. Symmetrized inner products of representations

Let G denote a group, F a representation of G over a field K, n a natural number and H a subgroup of the symmetric group S_n on $\mathbf{N}_n := \{1, ..., n\}$.

We form the n-fold inner tensor product $\otimes^n F$ of F with itself:

$$\overset{n}{\otimes} F := F \otimes \;\dots\; \otimes F \qquad (n \text{ factors}).$$

I.e., if $\mathbf{F}$ is a corresponding matrix representation, a realization of F by a matrix group, then the representing matrices of $\otimes^n \mathbf{F}$ are, for all $g \in G$,

$$\overset{n}{\otimes} \mathbf{F}(g) := \mathbf{F}(g) \times \dots \times \mathbf{F}(g) \qquad \text{(Kronecker product).}$$

Especially for the applications of representation theory to physical problems it is important to get some information on how $\otimes^n F$ splits.

A famous and very useful result due mainly to Schur, Weyl and van der Waerden (cf. [19, 20, 23–25] and [22], see also [3, § 67], [2, V and VI]) is that if K is algebraically closed and if its characteristic does not divide the order $|H|$ of $H \leqslant S_n$, then for certain irreducible representations D^i of H over K there is an in general reducible direct summand $F \boxminus D^i$ of $\otimes^n F$ which occurs with multiplicity f^i, the dimension of D^i:

$$(1.1) \qquad \overset{n}{\otimes} F = \sum_i{}' f^i (F \boxdot D^i)$$

(the prime indicates that the sum is taken over the indices i of certain irreducible K-representations D^i of H only).

The derivation of (1.1) for the cases $G := \mathrm{GL}(m, \mathbf{C})$, $F := \mathrm{id}_G$, $H := S_n$, $K := \mathbf{C}$ is the famous discovery of Schur, Weyl and van der Waerden throwing a light on the close connection between the representation theories of full linear and symmetric groups. And this special case is just the point on which the main interest of physicists in applications of the representation theory of symmetric groups is directed (cf., e.g., [5, 6, 26]). For (1.1) allows one to use the permutational symmetry say of a many-particle-system.

The permutational symmetry of a system of n particles of the same kind means that the symmetry group of the system is a wreath product of the form $G \sim S_n$. Indeed, a derivation of (1.1) using representations of wreath products needs only standard results of representation theory and hence it is easier to understand than the classical ones following the lines of Schur, Weyl and van der Waerden. We shall describe this now.

The *wreath product* $G \sim H$ of the group G with the permutation group $H \leqslant S_n$ on $\mathbf{N}_n = \{1, ..., n\}$ consists of the set

$$G \sim H := \{(f; \pi) \mid f : \mathbf{N}_n \to G,\ \pi \in H\},$$

together with the composition law

$$(f; \pi)(f'; \pi') := (ff'_\pi; \pi\pi')$$

(for all $i \in \mathbf{N}_n$, $\pi\pi'(i) := \pi(\pi'(i))$, $ff'_\pi(i) := f(i) f'_\pi(i)$, $f'_\pi(i) := f'(\pi^{-1}(i))$).

The normal subgroup

$$G^* := \{(f; 1_H) \mid f : \mathbf{N}_n \to G\} \trianglelefteq G \sim H$$

is called the *basis group* of $G \sim H$ and it is an inner direct product of n copies G_i of G:

$$G^* = \mathop{\times}_{i=1}^{n} G_i\,, \qquad G_i := \{(f\,;1_H) \mid \forall\, j \neq i\,, f(j) = 1_G\} \simeq G.$$

The basis group G^* possesses a complement H' which is isomorphic to H:

$$H' := \{(e\,;\pi) \mid \pi \in H\,, \forall\; i,\, e(i) := 1_G\} \simeq H.$$

The *diagonal subgroup* of G^*:

$$\operatorname{diag} G^* := \{(f\,;1_H) \mid f \text{ constant}\} \simeq G$$

is of great interest as well. Displaying the values of the functions, the elements of diag G^* are of the form

$$(f;1_H) = (f(1), \ldots, f(n);1_H) = (g, \ldots, g;1_H)\,,$$

so that

$$\operatorname{diag} G^* = \{(g, \ldots, g;1_H) \mid g \in G\}$$

and

$$(\operatorname{diag} G^*)H' = \{(g, \ldots, g;\pi) \mid g \in G,\, \pi \in H\} \simeq G \times H\,.$$

If we are given a representation F of G, we are interested in a representation of $G \sim H$ which yields $\otimes^n F$ when restricted to the diagonal subgroup of the basis group.

Let F be afforded by the left G-module M with underlying vector space V over the field K and operation $gv := F(g)v$, for all $v \in V$, $g \in G$. Then we obtain from the n-fold tensor product $\otimes^n V := V \otimes_K \ldots \otimes_K V$ (n factors) of V with itself a left G^*-module $\#^n M$ by defining

(1.2) $$(f;1_H)v_1 \otimes \ldots \otimes v_n := f(1)v_1 \otimes \ldots \otimes f(n)v_n\,.$$

Let us denote by $\#^n F$ the afforded representation. Its representing matrices are

$$\overset{n}{\#}\, \mathbf{F}(f;1_H) := \mathbf{F}(f(1)) \times \ldots \times \mathbf{F}(f(n))\,,$$

if $\mathbf{F}$ is again a matrix representation corresponding to F. We notice that

$$\overset{n}{\otimes}\, F = \overset{n}{\#}\, F \downarrow \operatorname{diag} G^*$$

is valid for the restriction $\#^n F \downarrow \operatorname{diag} G^*$ of $\#^n F$ to the diagonal subgroup diag G^* of the basis group.

The point is now that this representation $\#^n F$ resp. the module $\#^n M$ can be extended easily to a representation $\widetilde{\#}^n F$ of $G \sim H$ resp. to a left $G \sim H$ module $\widetilde{\#}^n M$ by extending the operation on $\otimes^n V$ to $G \sim H$ as follows:

$$(1.3) \qquad (f;\pi)v_1 \otimes \dots \otimes v_n := f(1)v_{\pi^{-1}(1)} \otimes \dots \otimes f(n)v_{\pi^{-1}(n)} .$$

This means for the representing matrices that we obtain $\widetilde{\#}^n \mathbf{F}(f;\pi)$ from $\#^n \mathbf{F}(f;1_H)$ by a suitable permutation of the columns: if $\mathbf{F}(g) = (f_{ik}(g))$, for all $g \in G$, and if

$$(1.4) \qquad \overset{n}{\#}\, \mathbf{F}(f;1_H) = \mathop{\mathsf{X}}_{i=1}^{n} \mathbf{F}(f(i)) = (f_{\alpha_1\beta_1}(f(1)) \dots f_{\alpha_n\beta_n}(f(n))) ,$$

we need only take

$$(1.5) \qquad \overset{\tilde{n}}{\#}\, \mathbf{F}(f;\pi) := (f_{\alpha_1\beta_{\pi^{-1}(1)}}(f(1)) \dots f_{\alpha_n\beta_{\pi^{-1}(n)}}(f(n))) .$$

We notice that

$$(1.6) \qquad \overset{n}{\otimes}\, F = \overset{n}{\#}\, F \downarrow \operatorname{diag} G^* = \overset{\tilde{n}}{\#}\, F \downarrow \operatorname{diag} G^* .$$

We are now in a position to derive (1.1) very easily. We consider elements $(f;\pi) \in (\operatorname{diag} G^*)H'$, i.e. elements $(f;\pi)$, where f is constant, say for all $i \in \mathbf{N}_n$, $f(i) = g \in G$.

The composition law tells us that for these elements the following is valid:

$$\begin{aligned}(f;\pi) &= (f(1), \dots, f(n);\pi) = (g, \dots, g;\pi) \\ &= (g, \dots, g;1_H)(1_G, \dots, 1_G;\pi) \\ &= (1_G, \dots, 1_G;\pi)(g, \dots, g;1_H) .\end{aligned}$$

This implies for the representation $\widetilde{\#}^n F$ of $G \sim H$:

$$\begin{aligned}(1.7) \qquad \overset{\tilde{n}}{\#} F(g, \dots, g;\pi) &= \overset{\tilde{n}}{\#} F(g, \dots, g;1_H)\, \overset{\tilde{n}}{\#} F(1_G, \dots, 1_G;\pi) \\ &= \overset{\tilde{n}}{\#} F(1_G, \dots, 1_G;\pi)\, \overset{\tilde{n}}{\#} F(g, \dots, g;1_H) .\end{aligned}$$

Let us now set out the following two notations:

$$\forall g \in G, \quad \hat{F}(g) := \overset{\tilde{n}}{\#} F(g, \dots, g;1_H) ,$$

$$\forall \pi \in H, \quad \check{F}(\pi) := \overset{\tilde{n}}{\#} F(1_G, \dots, 1_G;\pi) .$$

It is obvious that $\hat{F} : g \mapsto \hat{F}(g)$ resp. $\check{F} : \pi \mapsto \check{F}(\pi)$ are representations of G and H, respectively.

(1.7) leads to the crucial fact that for corresponding matrix representations we have:

$$(1.8) \qquad \forall g \in G,\ \pi \in H,\ \ \hat{\mathbf{F}}(g)\,\check{F}(\pi) = \check{\mathbf{F}}(\pi)\ \hat{\mathbf{F}}(g)\,.$$

Assuming now that K is algebraically closed and that char $K \dagger |H|$ we may apply an important result on commuting systems of matrices (see [2, I, § 8]) which follows from Schur's lemma: (1.8) implies that for each irreducible constituent D^i of $\check{F}$ there is a certain and in general reducible constituent of $\hat{F}$ – let us denote this constituent of $\hat{F}$ by $F \boxdot D^i$ and call it *the symmetrized inner product of F and D^i* (cf. the notation in [16]) – so that the following is valid:

$$(1.9) \qquad \overset{n}{\tilde{\#}}\, \mathbf{F}(g, \ldots, g;\pi) = \hat{\mathbf{F}}(g)\,\check{\mathbf{F}}(\pi) = \dot{+}'_{i}\, (\mathbf{F} \boxdot \mathbf{D}^i(g) \times D^i(\pi))\,.$$

(The prime indicates that the sum is taken over the indices i of a complete set of pairwise inequivalent irreducible constituents D^i of $\check{F}$ only instead of all the irreducible K-representations of H.) The dimension of $F \boxdot D^i$ is just the multiplicity of D^i in $\check{F}$. The desired result (1.1) is a corollary of (1.9) : (1.9) implies (use (1.6)) that

$$(1.10) \qquad \overset{n}{\otimes}\ \mathbf{F}(g) = \overset{n}{\tilde{\#}}\, \mathbf{F}(g, \ldots, g;1_H) = \dot{+}'_{i}\, (\mathbf{F} \boxdot \mathbf{D}^i(g) \times E_{fi})$$
$$= \dot{+}'_{i}\, f^i(\mathbf{F} \boxdot \mathbf{D}^i(g))\,.$$

That these symmetrized inner products $F \boxdot D^i$ are even more closely related to certain representations of the wreath product $G \sim H$ will turn out when we have derived their characters. This will be done in the next section.

2. The corresponding characters

Let us first evaluate the character of $\tilde{\#}^n F$. If $\{e_1, \ldots, e_r\}$ is a K-basis of V, the representation space of F, then $\{e_{i_1} \otimes \ldots \otimes e_{i_n} \mid 1 \leqslant i_\nu \leqslant r\}$ is a K-basis of $\otimes^n V$. Since

$$(f;\pi)e_{i_1} \otimes \ldots \otimes e_{i_n} = f(1)e_{i_{\pi^{-1}(1)}} \otimes \ldots \otimes f(n)e_{i_{\pi^{-1}(n)}}$$

$$= \Big(\sum_{\nu} f_{\nu i_{\pi^{-1}(1)}}(f(1))e_\nu\Big) \otimes \ldots \otimes \Big(\sum_{\mu} f_{\mu i_{\pi^{-1}(n)}}(f(n))e_\mu\Big) ,$$

the trace $\chi^{\widetilde{\#}^n F}(f;\pi)$ of this linear transformation on $\otimes^n V$ is

$$\chi^{\widetilde{\#}^n F}(f;\pi) = \sum_{(i_1,\ldots,i_n)} f_{i_1 i_{\pi^{-1}(1)}}(f(1)) \ldots f_{i_n i_{\pi^{-1}(n)}}(f(n)) .$$

This yields for the special case $\pi := (1\ 2\ \ldots\ n)$:

$$\begin{aligned} \chi^{\widetilde{\#}^n F}(f;(1\ \ldots\ n)) &= \sum_{(i_1,\ldots,i_n)} f_{i_1 i_n}(f(1)) \ldots f_{i_n i_{n-1}}(f(n)) \\ &= \operatorname{trace} F(f(1)\, f(n)\, f(n-1) \ldots f(2)) \\ &= \chi^F(f(1)\, f(\pi^{-1}(1)) \ldots f(\pi^{-n+1}(1))) , \end{aligned}$$

where χ^F denotes the character of F.

If π consists of several cycles $(i\ \pi(i)\ \ldots\ \pi^{k-1}(i))$, we have to form the analogous expression

$$(2.1) \qquad \chi^F(f(i)\, f(\pi^{-1}(i)) \ldots f(\pi^{-k+1}(i)))$$

for each of these cyclic factors of π and to multiply all these terms together in order to obtain $\chi^{\widetilde{\#}^n F}(f;\pi)$. We shall return to this general case in the next section.

Just as in the first section we consider again the elements $(f;\pi) \in (\operatorname{diag} G^*)H'$, i.e. the elements $(f;\pi) \in G \sim H$, where f is constant. For these elements we obtain from the considerations above:

$$(2.2) \qquad \chi^{\widetilde{\#}^n F}(g, \ldots, g;\pi) = \prod_{k=1}^{n} \chi^F(g^k)^{a_k(\pi)} ,$$

where $\pi = (a_1(\pi), \ldots, a_n(\pi))$ is the cycle-type of π, i.e. $a_k(\pi)$ $(1 \leqslant k \leqslant n)$ is the number of k-cycles in π with respect to the usual cycle-notation.

Before applying this to (1.9) in order to obtain the character of $F \boxdot D^i$, it seems to be useful to answer the following question which remained open in the first section: Which symmetrized inner products $F \boxdot D^i$ do actually exist (if F, H and K are given)?

We know that $F \boxdot D^i$ exists if and only if D^i is an irreducible constituent of $\check{F}$, where

$$\forall \pi \in H, \qquad \check{F}(\pi) := \overset{\sim}{\underset{}{\#}}{}^{n} F(1_G, ..., 1_G; \pi) .$$

The groundfield K is assumed to be algebraically closed and char $K = 0$.

Applying (2.2) we obtain for the multiplicity (F,D^i) of D^i in $\check{F}$:

$$\begin{aligned}(F,D^i) &= |H|^{-1} \sum_{\pi \in H} \chi^{\tilde{\#}^n F}(1_G, ..., 1_G; \pi)\, \chi^{D^i}(\pi^{-1}) \\ &= |H|^{-1} \sum_{\pi \in H} \chi^{D^i}(\pi^{-1}) \prod_{k=1}^{n} \chi^F(1_G)^{a_k(\pi)} \\ &= |H|^{-1} \sum_{\pi \in H} \chi^{D^i}(\pi^{-1})\, \chi^F(1_G)^{\Sigma a_k(\pi)} .\end{aligned}$$

(It seems remarkable that the last expression is a nonnegative integer since it is a multiplicity. This and related results will be considered in the last section.)

Hence denoting by f^F the dimension of F and by $c(\pi) := \Sigma a_k(\pi)$ the number of cycles of which π consists, we obtain the following necessary and sufficient condition for the existence of $F \boxdot D^i$:

(2.3) *If F is a K-representation of a group G and D an irreducible K-representation of $H \leqslant S_n$ with character χ^D, K algebraically closed and* char $K = 0$, *then the inner symmetrized product $F \boxdot D$ of F and D exists if and only if*

$$\sum_{\pi \in H} \chi^D(\pi)(f^F)^{c(\pi)} \neq 0,$$

where f^F denotes the dimension of F.

Let us now evaluate the character of $F \boxdot D^i$. An application of (2.2) to (1.9) yields

$$(2.4) \qquad \prod_k \chi^F(g^k)^{a_k(\pi)} = \sum_i{}' \chi^{F \boxdot D^i}(g)\, \chi^{D^i}(\pi) .$$

Since we have assumed K to be algebraically closed and char $K \dagger |H|$ we have the orthogonality relations at hand so that by multiplying (2.4) by $\chi^{D^j}(\pi^{-1})$ and summing over all $\pi \in H$ we get

$$(2.5) \qquad \chi^{F \boxdot D^i}(g) = |H|^{-1} \sum_{\pi \in H} \chi^{D^i}(\pi) \prod_{k=1}^{n} \chi^F(g^k)^{a_k(\pi)} .$$

This is the desired formula for the character of $F \boxdot D^i$.

Special cases of (2.5) are well-known and frequently used by physicists e.g. in the theory of many-particle-systems, where there is a permutational symmetry of the system, i.e. where $H = S_n$. G is taken to be a full linear group, say $G := \mathrm{GL}(m,C)$. For the representation F of G the identity map is taken $F := \mathrm{id}_{\mathrm{GL}(m,\mathrm{C})}$, and for D^i an ordinary irreducible representation $[\alpha]$ of S_n is taken, which corresponds to a partition α of n ($\alpha = (\alpha_1, ..., \alpha_h)$, $\alpha_i \in \mathrm{N}$, $\alpha_i \geqslant \alpha_{i+1}$, $\Sigma\alpha_i = n$) in a well-known manner (cf., e.g., [2, IV]). Applying (2.5) we obtain

$$\chi^{F \boxdot [\alpha]}(g) = n!^{-1} \sum_{\pi \in S_n} \chi^{\alpha}(\pi) \prod_{k=1}^{n} (\mathrm{trace}(g^k))^{a_k(\pi)} ,$$

if χ^α denotes the character of $[\alpha]$.

Now we use the fact (cf. [2, IV, theorem 1.2]) that we obtain $\chi^{F \boxdot [\alpha]}(g)$ by substituting the eigenvalues of the matrix $g \in \mathrm{GL}(m,C)$ for the variables of the S-function $\{\alpha\}$, where

$$\{\alpha\} := \det(x_i^{\alpha_j + m - j})/\det(x_i^{m-j})$$

($\alpha = (\alpha_1, ..., \alpha_h)$, $h \leqslant m$).

Besides this we use the trivial fact that $\mathrm{trace}(g^k)$ can be obtained by substituting the eigenvalues of g for the indeterminates of $\sigma_k := \Sigma_{i=1}^m x_i^k$. Applying this we obtain the famous result of Frobenius which connects the characters of irreducible representations of the full linear and symmetric groups (cf. [2], VI, (2.4)) as a corollary of (2.5):

$$(2.6) \qquad \{\alpha\} = n!^{-1} \sum_{\pi \in S_n} \chi^{\alpha}(\pi) \prod_{k=1}^{n} (\sigma_k)^{a_k(\pi)}$$

(Frobenius' formula).

Let us now return to the general case (2.5). It is of interest that this character is actually the character of a certain representation of $KG \sim H$, the group algebra of $G \sim H$ over K. For if we denote by $(F;D^i)$ the following representation of $G \sim H$:

$$(F;D^i) := \overset{\tilde{n}}{\#}\ F \otimes D^{i'} ,$$

where $D^{i'}(f;\pi) := D^i(\pi)$, then (2.2) and (2.5) yield

(2.7) $$\chi^{F \boxdot D^i}(g) = \chi^{(F;D^i)}\left(g, \ldots, g\,; |H|^{-1} \sum_{\pi \in H} \pi\right).$$

Thus we have described the characters of symmetrized inner products of representations in terms of characters of wreath products.

It seems worthwhile to compare (2.7) with the expression for *symmetrized outer products* $F \odot D$ of representations F of $G \leqslant S_m$ and D of $H \leqslant S_n$:

(2.8) $$F \odot D = (F;D) \uparrow S_{mn}$$

(cf. [10, 5.26]; "$\uparrow$" denotes induction of representations). It should be remarked, that (2.4) cannot be sharpened to $F \boxdot D^i(g) = (F\,; D^i)(g, \ldots, g\,; |H|^{-1} \Sigma\pi)$, say, since these two systems of matrices are in general of different dimension.

The symmetrized outer product is important since it satisfies an equation quite analogous to (1.1):

(2.9) $$\overset{n}{\#} F \uparrow S_{mn} = \sum_i f^i (F \odot D^i)\,,$$

where the sum is taken over all the irreducible K-representations D^i of H.

3. Irreducible characters of wreath products

We would like to sharpen the preceding results in order to give some information about the irreducible characters of $G \sim H$ over an algebraically closed field K. G is assumed to be a finite group now.

Let $\{F^1, \ldots, F^s\}$ denote a complete system of pairwise inequivalent and irreducible K-representations of G. Let V^i resp. M^i denote the representation space resp. the representation module of F^i, for $1 \leqslant i \leqslant s$.

Since G is assumed to be finite and K to be algebraically closed, a complete system of irreducible K-representations of G^* resp. of irreducible left KG^*-modules is given by the outer tensor products

$$F^* := F_1 \# \ldots \# F_n\,, \qquad F_i \in \{F^1, \ldots, F^s\}$$

resp. by

$$M^* := M_1 \# \ldots \# M_n, \qquad M_i \in \{M^1, \ldots, M^s\}\,,$$

where F^* is the representation afforded by M^* and M^* has $V_1 \oplus_K \ldots \oplus_K V_n$ (V_i the representation space of F_i) as underlying vector space and where the operation is defined by

$$(f;1_H)v_1 \otimes \dots \otimes v_n := f(1)v_1 \otimes \dots \otimes f(n)v_n \, .$$

If $\mathbf{F}_i$ is a realization of F_i by a matrix group, then $\mathbf{F}^*$, defined by

$$\mathbf{F}^*(f;1_H) = \mathbf{F}_1(f(1)) \times \dots \times \mathbf{F}_n(f(n)) \, ,$$

is a realization of F^*.

The inertia group $G \sim H_{F^*}$ of F^* is (see [10]):

$$(3.1) \qquad G \sim H_{F^*} = \{(f;\pi) \mid f : \mathrm{N}_n \to G, \forall i, F_i = F_{\pi(i)}\}$$
$$= G \sim (H \cap (S_{n_1} \times \dots \times S_{n_s})) \, ,$$

where S_{n_j} is the subgroup of permutations of S_n ($\geqslant H$) which permute only the indices i of factors F_i of F^*, which are equal to F^j, for $1 \leqslant j \leqslant s$.

In Section 1 we extended the representation $\#^n F$ of G^* to $G \sim H$, the inertia group of $\#^n F$, obtaining a representation $\widetilde{\#^n} F$ of $G \sim H$. Similarly we may extend $F^* = \#_i F_i$ to $G \sim H_{F^*}$ by applying the suitable column-permutations and obtaining an irreducible representation $\widetilde{F^*}$ of $G \sim H_{F^*}$.

Besides this, an irreducible K-representation D of $H \cap (S_{n_1} \times \dots \times S_{n_s})$ yields an irreducible K-representation D' of $G \sim H_{F^*}$ if we define

$$\forall (f;\pi) \in G \sim H_{F^*} \, , \qquad D'(f;\pi) := D(\pi) \, .$$

Clifford's theory of representations of groups with normal subgroups yields the fundamental result, that each irreducible K-representation of $G \sim H$ is of the form

$$(3.2) \qquad (\widetilde{F^*} \otimes D') \uparrow G \sim H \, ,$$

where the irreducible K-representations F^* and D' of $G \sim H_{F^*}$ are as described above (for more details cf. [10]).

(3.2) allows evaluation of the representing matrices, if F^* and D are known and hence this formula yields also the characters in a certain sense. Of course this method of evaluating the characters is by no means elegant and it would be nice to get some simpler methods to attack this problem (some character tables are known: the tables of $S_3 \sim S_2$, $S_4 \sim S_2$, $S_2 \sim S_4$ and $S_3 \sim S_3$ can be found in [12], the table of $S_2 \sim S_3$ in [15]; a complete description of the tables of $S_m \sim S_n$ for $mn \leqslant 10$, m and $n \neq 1$, is given in [18]).

There is no explicit formula available for the characters of (3.2) in the general case, there are some formulae for special cases only (see [10, 5.28, 5.29]). But the results of Section 2 allow the giving of at least the character of $\widetilde{F}^{*}\otimes D'$ so that no matrix has to be evaluated since we have only to carry out the inducing process.

To give this expression we use some abbreviations: If $(f;\pi)\in G \sim H$, we denote the cyclic factors of π by

$$(j_\nu\, \pi(j_\nu) \ldots \pi^{k_\nu-1}(j_\nu))\,, \qquad 1 \leqslant \nu \leqslant c(\pi)\,,$$

assuming that in each cycle the symbol j_ν is the least one which is involved and assuming that

$$j_1 \leqslant j_2 \leqslant \ldots \leqslant j_{c(\pi)}\,,$$

so that the j_ν are uniquely determined.

With each of these cycles we associate its *cycleproduct* g_ν with respect to f:

$$(3.3) \qquad g_\nu := f(j_\nu)\, f(\pi^{-1}(j_\nu)) \ldots f(\pi^{-k_\nu+1}(j_\nu)) \in G\,.$$

If $(f;\pi)\in G \sim H_{F^*}$, $F^* = \# F_i$, then (3.1) implies that

$$(3.4) \qquad F_{j_\nu} = F_{\pi^{-1}(j_\nu)} = \ldots = F_{\pi^{-k_\nu+1}(j_\nu)}\,.$$

We are now in a position to apply the results of Section 2: In order to obtain the character $\chi^{\widetilde{F}^*}(f;\pi)$ of $\widetilde{F}^*$ at $(f;\pi)$, where

$$\pi = \prod_{\nu=1}^{c(\pi)} (j_\nu \ldots \pi^{k_\nu-1}(j_\nu))\,,$$

since (3.4) holds we need only multiply together the expressions (2.1) for the cycleproducts so that we get

$$(3.5) \qquad \chi^{\widetilde{F}^*}(f;\pi) = \prod_{\pi=1}^{c(\pi)} \chi^{F_{j_\nu}}(g_\nu)\,.$$

This has been stated by Klaiber ([11]) without proof. From (3.5) we obtain the desired result

$$(3.6) \qquad \forall (f;\pi)\in G \sim H_{F^*}\,, \quad \chi^{\widetilde{F}^*\otimes D'}(f;\pi) = \chi^{D}(\pi) \prod_{\nu=1}^{c(\pi)} \chi^{F_{j_\nu}}(g_\nu)\,.$$

4. Enumeration under group action

The basic problem of this theory is – abstractly speaking – to evaluate the number of equivalence classes or symmetry types of functions between two finite sets X and Y with respect to an equivalence relation which is defined in an appropriate manner using two permutation groups acting on X and Y. I.e. we are given two finite sets X and Y, two permutation groups $H \leqslant S_X$ (the symmetric group on X) and $G \leqslant S_Y$ (the symmetric group on Y) and an equivalence relation "$\sim$" on

$$Y^X := \{\varphi \mid \varphi : X \to Y\} .$$

We ask for the number $|Y^X/\sim|$ of equivalence classes of Y^X. The equivalence classes are called the *symmetry types* of functions from X to Y with respect to the equivalence relation "$\sim$".

The theory of such enumeration problems and related questions has been developed independently by Redfield [14] and Pólya [13] in 1927 and 1937. It has important applications to graph theory (cf., e.g., the introductions to this theory given by de Bruijn [4] and Harary [7]) and may be thought of as being an important part of combinatorics (cf., e.g., Berge [1]).

There are three different types of this basic problem, differing in the way we define the equivalence relation "$\sim$" on Y^X using the given groups $G \leqslant S_Y$ and $H \leqslant S_X$:

Type I. $\varphi \sim \psi$: $\exists \pi \in H, \forall x \in X, \quad \varphi(x) = \psi(\pi^{-1}(x))$;

Type II. $\varphi \sim \psi$: $\exists \pi \in H, g \in G, \forall x \in X, \quad \varphi(x) = g\psi(\pi^{-1}(x))$;

Type III. $\varphi \sim \psi$: $\exists \pi \in H, \forall x \in X, \exists g_x \in G, \quad \varphi(x) = g_x \psi(\pi^{-1}(x))$.

The question is, how we can get in each of these cases the number $|Y^X/\sim|$ of equivalence classes of Y^X.

Of course, type III is the most difficult and most general situation. It will turn out that if we have solved the problem of type III the two other problems can be solved by restriction.

The solutions of the problems of type I and II are known but hitherto there is no solution of problem III published. Illustrative examples fôr these problems I and II are the well-known necklace problems (cf. [7]).

The equivalence classes of Y^X under "$\sim$" are of course the orbits of the permutation group $P \leqslant S_{Y^X}$ corresponding to the equivalence relation and depending of course on G and H. We denote these permutation groups as follows:

Type I: $P = E^H$,

Type II: $P = G^H$ (called the *power group* of G and H),

Type III: $P = [G]^H$ (the *exponentiation group* of G and H).

We notice that

(4.1) $\quad E^H \leqslant G^H \leqslant [G]^H$ *and* $\quad G^H \simeq G\times H\,,\quad [G]^H \simeq G \sim H$

(these two isomorphisms will be verified later on). Hence we may use Burnside's lemma to evaluate the desired number $|Y^X/{\sim}|$ of orbits of P on Y^X.

Burnside's lemma giving the number of orbits of a given permutation group P in terms of the cycle structure of the elements $p \in P$ reads as follows:

(4.2.) $\quad$ *no. of orbits of* $P = |P|^{-1} \sum_{p\in P} a_1(p)\,.$

It shows how we may proceed now and it gives us a hint how we may apply the representation theory of P in this context. For $a_1(p)$, the number of 1-cycles or fixed points of p, is just the value of the character χ^{NP} of the natural representation NP of P at p, where for a permutation group P on a finite set Ω, its *natural representation* NP is defined as follows:

(4.3) $\quad NP(p) : \mathbf{R}^{|\Omega|} \to \mathbf{R}^{|\Omega|} : (e_i \mapsto e_{p(i)},\ 1 \leqslant i \leqslant |\Omega|)\,,$

if $\{e_1, ..., e_{|\Omega|}\}$ is a basis of the $|\Omega|$-dimensional vector space $\mathbf{R}^{|\Omega|}$ over the real number field $\mathbf{R}$.

Since (4.1) holds we need only evaluate χ^{NP} for $P := [G]^H (\simeq G \sim H)$ and then sum up according to (4.2), which gives us the solution of problems of type III. The solutions of the other two problems will then be obtained by restriction, i.e. we have to sum up over the elements of the subgroup E^H resp. G^H only. Now we state (and then prove) in terms of Section 1 the crucial fact:

(4.4) $\quad$ *$[G]^H$ is a permutation group similar to the image of the representation $\widetilde{\#}^{|X|}\, NG$ of $G \sim H$.*

Proof. (i) Since (4.3) holds, the natural representation of $G \leqslant S_Y$ is defined as follows:

$$NG(g) : \mathbf{R}^{|Y|} \to \mathbf{R}^{|Y|} : (e_i \mapsto e_{g(i)},\ 1 \leqslant i \leqslant |Y|)\,.$$

Hence following Section 1, we obtain

$$\overset{|\tilde{X}|}{\#} NG(f;\pi) : \overset{|X|}{\otimes} \mathrm{R}^{|Y|} \to \overset{|X|}{\otimes} \mathrm{R}^{|Y|} : (e_{i_1} \otimes \ldots \otimes e_{i_{|X|}}$$

$$\mapsto f(1)\, e_{i_{\pi^{-1}(1)}} \otimes \ldots \otimes f(|X|)\, e_{i_{\pi^{-1}(|X|)}},\ 1 \leqslant i_\nu \leqslant |Y|)\,,$$

and that

$$f(1)\, e_{i_{\pi^{-1}(1)}} \otimes \ldots \otimes f(|X|)\, e_{i_{\pi^{-1}(|X|)}} =$$

$$= e_{f(1) i_{\pi^{-1}(1)}} \otimes \ldots \otimes e_{f(|X|) i_{\pi^{-1}(|X|)}}\,.$$

(Without loss of generality we may assume $X = \{1, \ldots, |X|\}$.) Hence

$$\overset{|\tilde{X}|}{\#} NG(f;\pi) : (e_{i_1} \otimes \ldots \otimes e_{i_{|X|}} \tag{4.5}$$

$$\mapsto e_{f(1) i_{\pi^{-1}(1)}} \otimes \ldots \otimes e_{f(|X|) i_{\pi^{-1}(|X|)}},\ 1 \leqslant i_\nu \leqslant |Y|).$$

Thus the image of $\tilde{\#}^{|X|}NG$ is a permutation group acting on the basis $\{e_{i_1} \otimes \ldots \otimes e_{i_{|X|}} \mid 1 \leqslant i_\nu \leqslant |Y|\}$ of $\otimes^{|X|} R^{|Y|}$.

(ii) To show that $[G]^H$ is similar (cf. [9, I, 5.7]) to this image of $\tilde{\#}^{|X|}NG$ since (i) holds, we need only give a bijection

$$\epsilon : Y^X \rightarrowtail\!\!\!\!\twoheadrightarrow \{e_{i_1} \otimes \ldots \otimes e_{i_{|X|}} \mid 1 \leqslant i_\nu \leqslant |Y|\}$$

and an isomorphism

$$\lambda : [G]^H \simeq \mathrm{Im}\left(\overset{|\tilde{X}|}{\#} NG\right)$$

so that

$$\lambda((g_1, \ldots, g_{|X|};\pi))(\epsilon(\psi)) = \epsilon(\varphi)\,, \tag{4.6}$$

where for $\psi \in Y^X$ the mapping $\varphi \in Y^X$ is defined by

$$\forall x \in X, \quad \varphi(x) := g_x \psi(\pi^{-1}(x))\,.$$

(iii) It is very easy to see that λ defined by

$$\lambda((g_1, \ldots, g_{|X|};\pi)) := \overset{|\tilde{X}|}{\#} NG(g_1, \ldots, g_{|X|};\pi)$$

is an isomorphism and that ϵ defined by

$$\epsilon(\varphi) := e_{\varphi(1)} \otimes \ldots \otimes e_{\varphi(|X|)}$$

is a bijection.

(iv) Having thus defined ϵ and λ we obtain from (4.5) that (4.6) is satisfied.

Applying (4.4) the results of Sections 2 and 3 especially (3.5) yield at once the following theorem solving the enumeration problem of type III, where the exponentiation group is the permutation group corresponding to the equivalence relation:

(4.7) *(Exponentiation group enumeration theorem) The number of orbits of* $[G]^H$ *is*

$$\frac{1}{|G|^{|X|}|H|}\sum_{(f;\pi)\in G\sim H}\prod_{\nu=1}^{c(\pi)}\chi^{NG}(g_\nu)=$$

$$=\frac{1}{|G|^{|X|}|H|}\sum_{(f;\pi)\in G\sim H}\prod_{\nu=1}^{c(\pi)}a_1(g_\nu)\,,$$

where (cf. (3.3)) if $\pi=\Pi_\nu(j_\nu\ldots\pi^{k_\nu-1}(j_\nu))$, *the* g_ν *are defined by* $g_\nu:=f(j_\nu)\,f(\pi^{-1}(j_\nu))\ldots f(\pi^{-k_\nu+1}(j_\nu))$.

Restriction to G^H yields (use (2.2)) a result of Harary and Palmer (cf. [8]):

(4.8) *(Power group enumeration theorem) The number of orbits of* G^H *is*

$$\frac{1}{|G|\,|H|}\sum_{(g,\pi)\in G\times H}\prod_{k=1}^{|X|}a_1(g^k)^{a_k(\pi)}=\frac{1}{|G|\,|H|}\sum_{(g,\pi)\in G\times H}\prod_{k=1}^{|X|}\Big(\sum_{i|k}ia_i(g)\Big)^{a_k(\pi)}.$$

For $\Sigma_{i|k}ia_i(g)$ is the number of 1-cycles of g^k. Restricting further down to E^H we obtain (Pólya):

(4.9) *The number of orbits of* E^H *is*

$$|H|^{-1}\sum_{\pi\in H}|Y|^{c(\pi)}\,.$$

5. Certain polynomials

As has been mentioned, the results of the preceding section provide a solution only of a basic problem since they give only the number of all the equivalence classes of Y^X.

Applications of the theory of enumeration under group action need often a finer examination of the situation described by these problems of types I, II and III. In many cases we want to evaluate not only the whole number of equivalence classes but even the number of equivalence classes which have special properties.

As can be seen from the cited literature, an important tool to attack such finer problems is the consideration of the so-called *cycle-index* of the involved permutation group. If the permutation group H is a subgroup of S_n and if $x_1, ..., x_n$ are independent indeterminates, the cycle-index $Z(H; x_1, ..., x_n)$ of H is the following polynomial over the rational field $\mathbf{Q}$:

$$(5.1) \qquad Z(H; x_1, ..., x_n) := |H|^{-1} \sum_{\pi \in H} x_1^{a_1(\pi)} \dots x_n^{a_n(\pi)}$$

$$= |H|^{-1} \sum_{\pi \in H} \prod_{k=1}^{n} x_k^{a_k(\pi)} \in \mathbf{Q}[x_1, ..., x_n] \; .$$

E.g.,

$$Z(S_2; x_1, x_2) = 2^{-1}(x_1^2 + x_2) \; .$$

By substituting 1 for the variable x_1, 0 for x_2, we obtain the value 2^{-1}. Polynomials over $\mathbf{Q}$ are called *numerical* if all their values over $\mathbf{Z}$ are in $\mathbf{Z}$. Hence for example $Z(S_2; x_1, x_2)$ is not numerical.

Snapper proved (cf. [21, theorem 1]) that the polynomial arising from a cycle index by substituting $\Sigma_{i|k}\, ix_i$ for the variable x_k is in fact numerical:

$$(5.2) \qquad q(H; x_1, ..., x_n) := |H|^{-1} \sum_{\pi \in H} \prod_{k=1}^{n} \Big(\sum_{i|k} ix_i \Big)^{a_k(\pi)}$$

is a numerical polynomial.

This result has been generalized by Rudvalis and Snapper (cf. [17, theorem 4.1]) as follows:

(5.3) *If* $\chi : H \to \mathbf{C}$ *is a generalized complex character of* H*, then*

$$N(H; x_1, ..., x_n) := |H|^{-1} \sum_{\pi \in H} \chi(\pi) \prod_{k=1}^{n} \Big(\sum_{i|k} ix_i \Big)^{a_k(\pi)}$$

is a numerical polynomial.

We would like to give a new proof of (5.3) using the results of the preceding sections and some steps of the original proofs of Snapper and Rudvalis.

The first step is to show that $N(H;x_1, ..., x_n) \in \mathbf{Q}[x_1, ..., x_n]$. In order to prove this, Rudvalis and Snapper noticed first, that for all cycle-types $(a_1, ..., a_n)$ occurring in H we have

$$\sum_{\pi \in H} \chi(\pi) \in \mathbf{Q}$$

if the sum is taken over all $\pi \in H$ with the same cycle-type $T_\pi = (a_1, ..., a_n)$.

To show that $N(H;x_1, ..., x_n)$ is in fact numerical, we need only prove that the values of $N(H;x_1, ..., x_n)$ are rational integers if $0 \leqslant x_i < |H|$, for the coefficients of the polynomial have $|H|$ as common denominator (see Snapper's paper). And that these special values are indeed rational integral now follows fairly easily from the results of Section 2: If $0 \leqslant x_i < |H|$, we consider an element $g \in S_m$, where

$$m := \sum_{i=1}^{n} ix_i \qquad \text{and} \qquad T_g = (x_1, ..., x_n)\,.$$

If NS_m denotes the natural representation of S_m and if D is an irreducible representation of H over $\mathbf{C}$, then we obtain from (2.5),

$$\begin{aligned}\chi^{NS_m \boxdot D}(g) &= |H|^{-1} \sum_{\pi \in H} \chi^D(\pi^{-1}) \prod_{k=1}^{n} \chi^{NS_m}(g^k)^{a_k(\pi)} \\ &= |H|^{-1} \sum_{\pi \in H} \chi^D(\pi^{-1}) \prod_{k=1}^{n} \Big(\sum_{i|k} ix_i\Big)^{a_k(\pi)},\end{aligned}$$

and this value is rational integral since it is a value of a character and rational or it is 0 in case that $NS_m \boxdot D$ does not exist.

Of course the same is valid if we consider instead of an irreducible character χ^D a generalized character χ and we are finished.

Added in proof. During the last three years, when this paper was already submitted, along the same lines of representation theory of wreath products the cycle index of $[G]^H$ as well as its enumeration theorem in weighted form was derived in the following papers: A. Kerber, Der Zykelindex der Exponentialgruppe, Mitt. Math. Sem. Giessen 98 (1973) 5–20; and W. Lehmann, Das Abzähltheorem der Exponentialgruppe in gewichteter Form, Mitt. Math. Sem. Giessen 112 (1974) 19–33.

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